Open circle maps: Small hole asymptotics

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Abstract

We consider escape from chaotic maps through a subset of phase space, the hole. Escape rates are known to be locally constant functions of the hole position and size. In spite of this, for the doubling map we can extend the current best result for small holes, a linear dependence on hole size h, to include a smooth $h^2 \ln h$ term and explicit fractal terms to h^2 and higher orders, confirmed by numerical simulations. For more general hole locations the asymptotic form depends on a dynamical Diophantine condition using periodic orbits ordered by stability.

1 Introduction

Here we consider open dynamical systems, in which motion is considered only until the trajectory reaches a specified subset of phase space, the "hole(s)". The initial conditions are distributed with respect to some measure, typically an invariant measure on the phase space of the corresponding closed system (the "escape" problem), or on such a measure restricted to hole(s) ("recurrence" or "transport" problems). There are also relevant questions where there are two systems connected by a hole (the "metastability" problem). There is a vast literature in mathematics and physics on open dynamical systems and applications, some of which is mentioned in [2, 8, 9].

For the escape problem, we expect that for strongly chaotic systems (eg with exponential decay of correlations), the survival probability $P(n) = \mu(M_n)$ decays exponentially with time n, so that an escape rate can be defined:

$$\gamma = \lim_{n\to\infty} -\frac{1}{n} \ln P(n) \tag{1}$$

Here μ is the measure of initial conditions (assumed to be invariant under the dynamics) and M_n the subset of the phase space M that survives for at least n iterations before reaching the hole $H \subset M$. In general the limit might not exist, and we need upper and lower escape rates, possibly infinite. Here we restrict to maps, considering only a discrete time n; similar questions apply to flows.

It is of interest to consider how the escape rate depends on the hole, in particular its size $h = \mu(H)$. On one hand, there have been attempts to quantify the small hole asymptotics of the escape rate, that is, its behaviour for a sequence of holes contracting to a point $x \in M$. Ref. [17] numerically observed the effects of periodic orbits on the escape rate from a chaotic map with a small hole. Ref. [5] expresses escape rates in terms of correlation functions, finding good numerical agreement in the case of diamond billiard with continuous time escape rate computed to order h^2; the heuristic arguments proposed there suggest that for billiards with holes typically considered (holes that allow particles to escape that reach a small region of the boundary, but from any direction) the first order result is impervious to a short periodic orbit located in the hole, but that corrections would be needed for the second order formulas. Ref. [14] shows rigorously that

$$\gamma = h(1 - \Lambda_x^{-1}) + o(h) \tag{2}$$

for a wide class of hyperbolic maps, where Λ_x is the stability eigenvalue of the periodic orbit starting from $x \in M$, infinite if the orbit of x is aperiodic. This result (amongst others) was also shown independently in a study of the open doubling map [6].

On the other hand it is known that the largest invariant set that never reaches the hole, a subset of M_∞, is a locally constant function of the hole. This is also mentioned in [14] but was noted much earlier [21]. It

then implies the escape rate, and also properties such as the relevant Lyapunov exponents and entropy, are locally constant. This can be seen (but was not remarked on) for the diamond billiard in Fig. 4 of Ref. [5]. While putting an apparent limit on small hole asymptotic results, this effect is also related to the very helpful observation that open dynamics restricted to the repeller is normally a subshift of finite type; almost always in the case of the doubling map considered here. For very recent results on the existence, continuity and locally constant properties of the escape rate for more general hyperbolic systems, see Ref. [7].

We note that the locally constant property does not exclude the possibility that the escape rate is smoother as $h \to 0$ than the single derivative required by Eq. (2). This paper is an attempt to push the small hole asymptotics as far as possible in the simplest possible dynamical system, the doubling map, with the hope of elucidating similar effects in more general contexts. Sec. 2 discusses previous work relevant to the doubling map. Sec. 3 considers a hole starting at zero and uncovers an explicit, but fractal, expansion for the escape rate to arbitrary order in h, including a smooth $h^2 \ln h$ term as the first contribution beyond linear. Sec. 4 considers arbitrary hole locations, finding numerical support for a general $h^2 \ln h$ term for rational locations, but rigorous arguments for terms as slow as $h/\ln h$ for some Liouville locations, ie very well approximable by rationals. In more general maps, the expansion appears to depend on an approximation theory using periodic orbits ordered by stability. Finally, Sec. 5 explores some connections with the Gauss map.

2 Existing results

The doubling map $x \to 2x \pmod 1$ has an invariant measure μ simply given by Lebesgue, so the hole size h is simply the length of the relevant interval(s). There is a correspondence between the binary representation of a point x and symbolic dynamics for the partition $\{[0, 1/2), [1/2, 1)\}$, modulo minor details to do with the dyadic rationals (ie fractions with denominator a power of two).

The case of a hole between 0 and h has received some attention in the literature, also including more general expanding circle maps [13, 21]. In brief, given a hole size h, find

$$h_- = \sup_{x \le h} \{2^n x > x \pmod 1, \quad \forall n \in \mathbb{N}\} \tag{3}$$

This is closely related to the concept of a "minimal prefix" [16]. The quantity h_- is almost always a dyadic rational $i/2^n$ for some integer i, and in this case h lies in the interval of constant escape rate $i/2^n \le h \le i/(2^n - 1)$, often abbreviated here as just "interval." Note that each periodic orbit of the map appears exactly once as the upper limit of one of these intervals.

Within an interval, the dynamics is a subshift of finite type. If we write the (n term) binary expansion of $1 - h$:

$$1 - h_- = \sum_{k=1}^{n} a_k 2^{-k} \tag{4}$$

then the leading eigenvalue of the transition matrix is $\beta/2$ where β is the solution of

$$1 = \sum_{k=1}^{n} a_k \beta^{-k} \tag{5}$$

with $\beta \approx 2$. Thus we find the escape rate

$$\gamma = -\ln \frac{\beta}{2} \tag{6}$$

Both the local dimension of the set of irrational h_- and the Hausdorff dimension of the relevant repelling set are equal to $\ln \beta / \ln 2 = 1 - \gamma / \ln 2$. For irrational h_- the escape rate is computed as in the rational case (but taking $n \to \infty$) since it is continuous and monotonic. Thus the case of $H = [0, h]$ is of interest for β-expansions.

3 Small hole asymptotics

We now consider the question as to what asymptotic results are possible in this case. The intervals of constant escape rate give an important constraint on the possible form of such asympototics: γ is constant,

but h has varied by an amount proportional to h^2 (roughly equal to h^2 when $i = 1$). Thus γh^{-2} has jumps of unit size occurring over a short distance relative to h itself: the $O(h^2)$ term in the expansion, if it exists, must have jump discontinuities.

For any h contained in an interval, we have $\gamma(h) = \gamma(h_-)$ so it is sufficient to consider just values h_- on the left of an interval. From the monotonicity and continuity we may furthermore calculate γ as a limit of rational or irrational h_- as convenient. For the following argument we use irrational $h = h_-$ and the binary expansion of h rather than the $1 - h$ above:

$$h = 2^{-J} K_0 = 2^{-J} \sum_{k=1}^{\infty} A_k 2^{-k} \tag{7}$$

Here, $J = (\ln K_0 - \ln h)/\ln 2$ is a non-negative integer so that $K_0 < 1$. There are at least two logical choices for J: We can set $1/2 \leq K_0 < 1$ so that K_0 (and each K_m below) is bounded above and below; this is most helpful in determining the magnitude of the relevant terms. The other choice is to set $J = 0$ which simplifies the computations. For now we need to keep J arbitrary. The other new quantities in the above expression are $A_k = 1 - a_{J+k}$, with all the non-positive A_k equal to zero. Thus the expansion for β becomes

$$1 = \sum_{k=1}^{\infty} a_k \beta^{-k} = (\beta - 1)^{-1} - \sum_{k=1}^{\infty} A_k \beta^{-J-k} \tag{8}$$

We know from [14] that to leading order $\gamma \approx h/2$, since the fixed point at $x = 0$ has $\Lambda = 2$. Thus we write

$$\beta = 2 - \sum_{l=1}^{\infty} \epsilon_l h^l \tag{9}$$

where $\epsilon_1 = 1$, and we allow the other ϵ_l to be functions $o(h^{-1})$ as $h \to 0$. Substituting for β, expanding both powers of β in a binomial expansion, substituting h/K_0 for 2^{-J} and equating coefficients with equal powers of h, we find that

$$\epsilon_2 = -1 + \frac{J}{2} + \frac{K_1}{2K_0} \tag{10}$$

$$\epsilon_3 = 1 + \frac{3J^2}{8} + \frac{J}{8}\left(\frac{6K_1}{K_0} - 11\right) + \frac{1}{8}\left(\frac{K_2}{K_0} + \frac{2K_1^2}{K_0^2} - \frac{11K_1}{K_0}\right) \tag{11}$$

and so on, where

$$K_m = \sum_{k=1}^{\infty} A_k k^m 2^{-k} \tag{12}$$

consistent with the definition of K_0 above. The K_m for $m > 0$ are fractal functions bounded on $[1/2, 1)$ as shown in Fig. 1; they are discontinuous at dyadic rationals.

We also see that at second order the expansion involves Jh^2 which gives rise to a smooth $h^2 \ln h$ term. Thus we can extend Ref. [14] result to

$$\gamma = \frac{h}{2} + \frac{h^2 |\ln h|}{4 \ln 2} + \frac{h^2}{4}\left(-\frac{3}{2} + \frac{\ln K_0}{\ln 2} + \frac{K_1}{K_0}\right) + \ldots \tag{13}$$

for h an irrational value in the fractal set not contained in any interval.

If h lies in an interval $[h_-, h_+] = [i/2^n, i/(2^n - 1)]$ we identify it with the discontinuity occurring at $K_0 = i/2^{(n-J)}$. The jump at this point is $K_1^- - K_1^+ = 2^{(1-(n-J))}$ independent of i, which leads to a cancellation to second order with the discrepancy in the first term, since $h_- - h_+ = i/2^{2n}$ to this order; all other terms are smooth. Thus, the jumps in γ are exactly those required by the locally constant intervals.

In what sense does this expression converge in the $h \to 0$ limit? Considering a fixed domain of K_0, intervals containing $2^{-J} K_0$ decrease with increasing J. Thus we expect convergence for almost all fixed $K_0 \in (1/2, 1)$ as $J \to \infty$, where $h = 2^{-J} K_0$, with an error of smaller order than the last retained term. On the other hand, there is no guarantee that for fixed J the terms with higher powers of h continue to decrease.

Recall that we did not specify J exactly; we can see that the transformation $J \to J + 1$ has the effects $K_0 \to K_0/2$ and $K_1 \to (K_1 + K_0)/2$, leaving the above expression for γ unaffected. But this also means

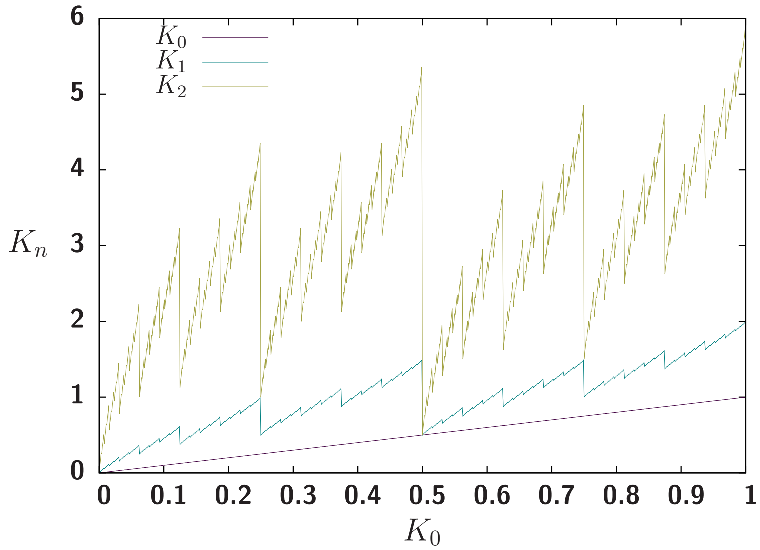

Figure 1: The K_m functions defined in Eq. (12).

that the coefficient of h^2 is log-periodic, ie a periodic function of $\ln h$. We can thus write the expansion of γ as a series involving h, $h^2 \ln h$ and $h^{2+2\pi iq/\ln 2}$ for $q \in \mathbb{Z}$, however the coefficients (Fourier expansion of the K functions in the variable $\ln h$) do not appear to have a simple analytic form. They are also slow to converge due to the discontinuities. We can continue analogously with h^3 and higher, indicating that the Mellin transform of γ has a semi-infinite periodic forest of poles reminiscent of those for Julia sets [3].

Using the arbitrariness in J, we now set $J = 0$ to yield the simplest form of these expressions: we can substitute h for K_0 and evaluate the other K functions at h, giving the main result:

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + O(h^4 \ln^3 h) \tag{14}$$

$$\gamma_1 = \frac{h}{2} \tag{15}$$

$$\gamma_2 = \frac{2K_1 h - 3h^2}{8} \tag{16}$$

$$\gamma_3 = \frac{6K_1^2 h + 3K_2 h^2 - 27K_1 h^2 + 14h^3}{48} \tag{17}$$

where we note that $K_m(h) \sim h|\ln^m h|/\ln^m 2$ as $h \to 0$.

Fig. 2 shows a comparison between the escape rate calculated directly and with the above formulas. The direct numerical computation uses a Markov matrix of side length $N = 2^{16}$ ($N = 2^{19}$ for $h < 0.01$) that does not need to be stored due to its regular structure. The leading eigenvalue corresponding to γ is obtained by repeated multiplication of the matrix on a vector with initially identical nonzero entries, using long double (19 digit) precision. The direct escape rate is plotted, then subtracting each of the γ_i in turn, leading to smaller and smaller remainders. We can see that the locally constant intervals (smooth curves in the direct computation) become smaller for smaller h, but remain as spikes at dyadic rationals. Elsewhere, the remainder decreases with the order of the approximation, showing good convergence despite the fractal character of the functions involved.

Various connections may be made between the K functions and self-affine functions and sets appearing in the literature. The graph of K_1 is easily seen to be a discontinuous self-affine set, see Ref. [19] for some relevant properties. It shares the relation

$$K_1(x/2) = (K_1(x) + x)/2 \tag{18}$$

with the Takagi function, used to study diffusion in extended doubling maps [15], but the similarity appears to end there, as the Takagi function is continuous. We can construct a generating function for all the K

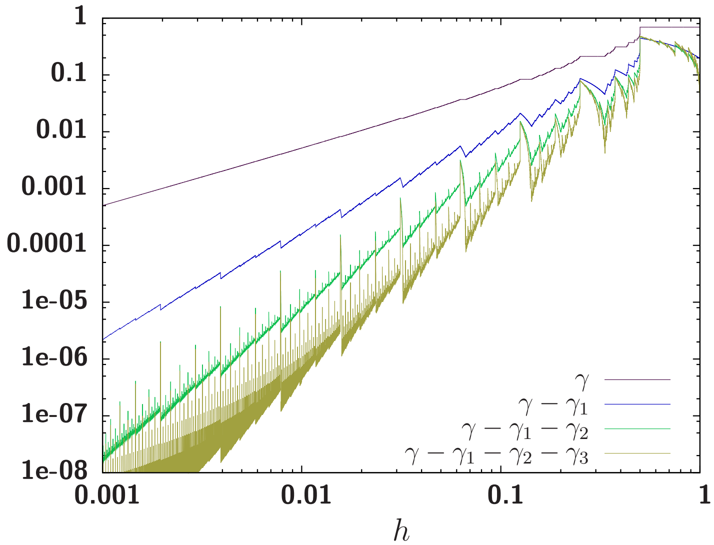

Figure 2: Numerical test of the small hole expansion for the escape rate, Eq. (14).

functions, (cf Eq. 5)

$$G_\beta(x)=\sum_{k=1}^{\infty}A_k\beta^{-k} \tag{19}$$

so that, at least formally, we have

$$K_n(x)=(-\beta\partial_\beta)^n G_\beta(x)|_{\beta=2} \tag{20}$$

The limits here are quite subtle, since $G_\beta(x)$ effectively computes the Bernoulli convolution corresponding to β, and it is known that the distribution of the latter is singular when β is Pisot, including a sequence accumulating at $\beta=2$ [20]; it might be interesting to investigate these connections further.

4 Dependence on hole position

For holes at general locations, $H=[H_1,H_2]$ with $H_2=H_1+h$ we no longer have an explicit formula for the characteristic polynomial (except for Markov holes; see [11]) so it is not yet possible to carry out the same expansion as above. However there are a few observations that can be made, which also apply to more general piecewise expanding maps.

Heuristic argument for $h^2\ln h$ There are reasons to believe the $h^2\ln h$ term is more widely applicable. In a general hyperbolic map, the linear dependence of the escape rate on the periodic orbit stability arises because that is the extent of the "shadow" [4] cast by the hole on itself under iteration around the periodic orbit. Beyond this, we need to study the shadow of the hole on itself along orbits that do not remain close to the short periodic orbit (if there is one) passing through the hole. The shortest of these are typically of length $|\ln h|$, with stabilities of order h^{-1}. This will make a piece of the hole of size roughly h^2 inaccessible to the escaping trajectories. At the next length there will be more periodic orbits, balanced by greater instabilities, also yielding of order h^2. This continues until double the original length, say $2|\ln h|$ beyond which the periodic orbits are effectively just shadowing combinations of shorter periodic orbits. From this argument we might expect these to give a total contribution to the escape rate of order $h^2\ln h$. However it will depend on the details of the periodic orbits. We now introduce a more systematic approach and show that reality is more subtle.

Relation to dynamical Diophantine properties Let us fix H_1 and take $h\to 0$. In general we expect that the escape rate is given (roughly) by

$$\gamma\approx h(1-\Lambda^{-1}) \tag{21}$$

where Λ is the stability eigenvalue of the shortest periodic orbit in the interval. The approximation arises from omitting the small h limit needed for Eq. (2). If H_1 is itself aperiodic, we have $\gamma/h \to 1$ as $h \to 0$, however a short periodic orbit x lying just above h_1 will lead to smaller γ when $h > x - H_1$; if there are infinitely many such orbits (where "short" is relative to an appropriate function of h), this may affect the limiting escape behaviour beyond linear order.

So, assume that H_1 is not itself periodic and consider a sequence x_n of periodic points approximating H_1 from above. We obtain a sequence

$$\gamma_n \approx h_n(1 - \Lambda_n^{-1}) \tag{22}$$

of approximate escape rates with hole sizes $h_n = x_n - H_1$. The correction to the linear term, $h_n\Lambda_n^{-1}$ is of an order of magnitude relative to h_n that depends on the stability of the periodic orbit.

(Pre-)periodic hole locations For a rational $H_1 = p/q$, that is, a periodic or preperiodic points for the doubling map, the distance to a rational approximation (ie periodic point) with denominator $2^n - 1$ cannot be smaller than $1/(q(2^n - 1))$, thus the size of a hole covering H_1 and its approximation is bounded by a known constant multiple of Λ^{-1} and so the correction term (making the same convergence assumption) is order h^2, hence not dominating the heuristic $h^2 \ln h$ term above. A symbolic version of the same argument could also extend to more general expanding maps. It may be possible to find explicit formulas for the characteristic polynomial in simple cases (piecewise linear map, short (pre)-periodic orbit), leading to the coefficient of $h^2 \ln h$ along the lines of the previous sections, however we will confine ourselves here to a numerical investigation of the doubling map.

We consider hole locations H_1 which are all the rational values with denominators in $\{1, 2, \ldots, 10, 15, 21, 31, 63\}$ thus including all periodic points with periods $p \leq 6$ and also 13 preimages of orbits with periods $p \leq 4$. The matrix size used was $N = 9999360 = 2^{10} \times 3^2 \times 5 \times 7 \times 31$ and the power method continued until a tolerance of 3×10^{-14} was reached, using 19 digit precision. For each location, the escape rate was computed for holes of size $h = 41/N$ and $h = 2^p 41/N$, corresponding to each other under the p times iterated map. The number 41 was chosen as having no common factor with the denominators, and roughly the smallest size consistent with the above tolerance. Removing the known linear term from the escape rates, these two points are then used to estimate $h^2 \ln h$ and h^2 terms corresponding to the sequence of hole sizes found by multiplying by powers of 2^p. As with the hole at zero discussed previously, the coefficient of the $h^2 \ln h$ term is found to be approximately independent of h, but the coefficient of h^2 is a fractal function of h.

The results, depicted in Fig. 3, are consistent with the formula

$$\gamma = (1 - \Lambda^{-1})h + \frac{(1 - \Lambda^{-1})^2}{\ln 2} h^2 |\ln h| + O(h^2) \tag{23}$$

where Λ is taken to be infinite for preperiodic points, and the $\ln 2$ is chosen for consistency with the previous case $H_1 = 0$. The single outlier near the top right of the figure corresponds to the point $H_1 = 62/63$, the period 6 orbit most affected by the fixed point at 1; here convergence of the escape rate to the small hole limit is slowest, so the extrapolation is not as accurate. In more general cases, note that where a periodic orbit has negative stability eigenvalue (as in the tent map for odd periods) Eq. (2), and hence its generalisation, need to be modified if the periodic point lies on the boundary of the hole.

Typical hole locations Typical behaviour is given by Borel-Cantelli results, which as discussed in Ref. [1] apply to the doubling map, and more generally. The relevant theorem states that an infinite number of the iterates $2^n x \pmod 1$ of the point x under the doubling map are contained in a shrinking sequence of intervals I_n for almost all x if and only if the sum of the measures (ie lengths) of the intervals diverges. For example, the intervals $[H_1 - (n \ln^k n)^{-1}, H_1]$ contain infinitely many $y_n = 2^n H_1 \pmod 1$ for almost every H_1 if and only if $k \leq 1$. Each such point leads to a periodic point $x_n = H_1 + (h_1 - y_n)/(2^n - 1)$ lying in $[H_1, H_1 + ((2^n - 1)n \ln^k n)^{-1}]$. Thus we expect an infinite sequence of holes h_n of size as low as $((2^n - 1)n \ln n)^{-1}$ with escape rate

$$\gamma_n \approx h_n(1 - 2^{-n}) \approx h_n(1 - h_n |\ln h_n| \ln |\ln h_n|) \tag{24}$$

This argument is not rigorous as it depends crucially on the convergence rate of Eq. (2) but suggests that for typical H_1 a bound on the correction term might need to be slightly greater than $|h^2 \ln h|$.

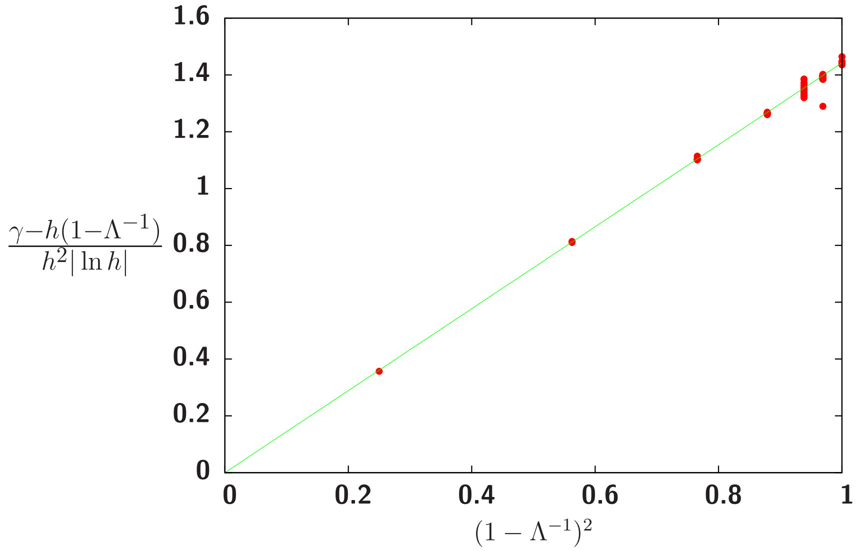

Figure 3: Numerical investigation of the dependence of the $h^2|\ln h|$ term for rational hole locations; see Eq. (23).

The well approximable case In contrast to the above heuristic and numerical arguments, the following negative result is rigorous:

Theorem 1. *Given any continuous strictly increasing function g defined on $[0, 1/3]$ with $g(0) = 0$ and $g(1/3) > 1/4$, there is an irrational (ie aperiodic) H_1 such that the doubling map with hole $[H_1, H_1 + h]$ has escape rate $\gamma(h)$ so that $\gamma(h)/h - 1$ approaches zero but is not $O(g(h))$ as $h \to 0$.*

Proof: Consider

$$x_K = 1 - \sum_{k=1}^{K} \frac{1}{2^{n_k} - 1} \tag{25}$$

for a strictly increasing sequence n_k with $n_1 = 2$. Clearly all the x_K lie in the unit interval. Then, we define

$$H_1 = \lim_{K\to\infty} x_K \tag{26}$$

and construct the sequence n_k so that H_1 satisfies the required property.

First, assume that n_k divides n_{k+1} for all k. Then, $2^{n_k} - 1$ divides $2^{n_{k+1}} - 1$ and so x_K is a rational with denominator $2^{n_K} - 1$ (or a factor) and hence a periodic orbit of length n_K (or a factor). Furthermore, since the sequence is strictly increasing, $n_{k+1} \geq 2n_k$ which implies that the limit H_1 is irrational as required.

There are two further lower bounds on the growth rate of n_k. One comes from the limit in Eq. (2): There exists a function $H(\epsilon)$ (implicitly depending on x_K) such that for all $h < H(\epsilon)$ we have

$$|\gamma(h)/h - (1 - 2^{-n_k})| < \epsilon \tag{27}$$

considering the hole $[x_K - h, x_K]$ shrinking to the periodic orbit x_K. We choose $\epsilon = 2^{-2n_k}$ to ensure that $\gamma(h)/h$ is sufficiently close to that determined from the periodic orbit. This requires $n_{k+1} > -\log_2 H(2^{-2n_k})+1$, where the final term ensures that the hole can extend downward beyond x_{K+1} to h_1.

Finally, we need to ensure that the correction 2^{-n_k} is greater than $g(h)$ for the hole $[h_1, x_K]$. This is accomplished by requiring $n_{k+1} > -\log_2 G^{-1}(2^{-n_k}) + 1$ where G^{-1} is the inverse of G, an arbitrarily chosen monotonic function with the same conditions as g but with $G(h) \neq O(g(h))$ as $h \to 0$. Thus any sequence satisfying n_k divides n_{k+1} and

$$n_{k+1} > \max[-\log_2 H(2^{-2n_k}) + 1, -\log_2 G^{-1}(2^{-n_k}) + 1] \tag{28}$$

satisfies the requirements of the theorem. $\square$

Examples: For $g(h) = h^\alpha$ with $0 < \alpha < 1$ we need at least $n_{k+1} > n_k/\alpha$. For $g(h) = -1/\ln h$ (so that $\gamma(h) = h + O(h/\ln h)$) we need at least $n_{k+1} > C2^{n_k}$, which has extremely rapid growth leading to H_1 as a Liouville number.

Remarks: The main idea of the above proof is to find aperiodic points well approximated by periodic orbits of the map, in particular such that the separation is much smaller than the inverse stability of the periodic orbit, Λ^{-1}. Approximation by periodic orbits has also appeared in different contexts, for example in obtaining the multifractal properties of Lyapunov exponents [18]. Ordering periodic orbits by stability is a more generally useful technique for systems with widely varying expansion rates [10].

5 Epilogue: The Gauss map

Finally, there is an intriguing connection with $h^2 \ln h$ in the literature, namely for the Gauss map $x \to \{1/x\}$ with invariant measure $d\mu = (\ln 2(1 + x))^{-1} dx$. The natural symbolic dynamics $x \to \lfloor 1/x \rfloor$ gives the partial quotients a_k of the continued fraction expansion $x = [0; a_1 a_2 \ldots]$. The periodic orbits are quadratic irrationals. Thus, a hole $H = [0, 1/(n+1)]$ corresponds to escape when $a_k > n$. The hole size is $h = \int_H d\mu = \ln(1 + (n+1)^{-1})/\ln 2$. Hensley [12] analyses this problem using the transfer operator

$$L_{s,n} f(t) = \sum_{k=1}^{n} \frac{f(1/(k+t))}{(k+t)^s} \tag{29}$$

giving its leading eigenvalue $\lambda(s, n)$ in his Eq. (7.9) as

$$\lambda\left(2 - \frac{\theta}{n}, n\right) = 1 + \left(\frac{\pi^2\theta - 12}{12 \ln 2}\right) \frac{1}{n} - \frac{\theta \ln n}{n^2 \ln 2} + O(n^{-2}) \tag{30}$$

Setting this equal to unity, he finds that the value of θ, and hence the Hausdorff dimension of the invariant set, to contain a term proportional to $\ln n/n^2$, ie also $h^2 \ln h$. The escape rate, given by $-\ln \lambda(2, n)$, has however no such term!

For more general hole positions, the previous discussion suggests that the asymptotics depends on the approximation of the hole position using periodic orbits ordered by stability. The Gauss map has infinitely many branches, however there are only finitely many periodic orbits with a given stability bound. It would thus be interesting to analyse the escape rate for general hole positions using approximation by quadratic irrationals.

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